

# Gauge-invariant and coordinate-independent perturbations of stellar collapse II: matching to the exterior

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In Paper I in this series we constructed evolution equations for the complete gauge-invariant linear perturbations of a time-dependent spherically symmetric perfect fluid spacetime. A key application of this formalism is the interior of a collapsing star. Here we derive boundary conditions at the surface of the star, matching the interior perturbations to the well-known perturbations of the vacuum Schwarzschild spacetime outside the star.

## I. INTRODUCTION AND OVERVIEW

In Paper I in this series [1] we constructed evolution equations for the complete gauge-invariant linear perturbations of a time-dependent spherically symmetric perfect fluid spacetime, with arbitrary two-parameter equation of state  $p = p(\rho, s)$ . We isolated true degrees of freedom for all perturbations. They obey a hyperbolic system of wave and transport equations. A key application of this formalism is the interior of a collapsing star. Here we derive boundary conditions at the surface of the star, matching the interior perturbations to the well-known perturbations of the vacuum Schwarzschild spacetime outside the star. Our formalism is now complete and ready for numerical work.

As in Paper I, we use the covariant and gauge-invariant perturbation formalism of Gerlach and Sengupta [2], and our notation is compatible with theirs. The combination of Paper I and the present Paper II is intended to be self-contained, but we refer the reader back to Paper I for some definitions and results. In our presentation, we go from the general to the specific.

In Section II we discuss continuity conditions across a hypersurface in spacetime where the stress-energy tensor is finite but possibly discontinuous. We argue that the appropriate choice of continuous perturbation fields are the perturbations of the 3-metric and extrinsic curvature of the surface with *contravariant* indices. We give those quantities in terms of metric perturbations and their first derivatives.

In Section III we then restrict the background spacetime to be spherically symmetric, but still allow for arbitrary matter, and for the matching surface to be either timelike or spacelike. We identify a complete set of continuous *gauge-invariant* perturbations in the notation of Paper I, separately for the axial and polar perturbations, decomposing all tensor quantities into components in a frame aligned with the surface. This simplifies and corrects results of [3,4].

In section IV we restrict consideration to perfect fluid matter and a timelike matching surface, namely the surface of the collapsing star, and state the matching conditions across the stellar surface in terms of variables specialized to fluid matter. The conditions for the axial perturbations are fairly simple.

In section V we bring the continuity conditions for the polar perturbations into a final form that shows clearly how information crosses the stellar surface in both directions (“extraction” and “injection”), in a way that is natural for a numerical evolution.

We now sketch the form our main results are going to take. In Paper I we characterized the *axial* perturbations in the interior by a tangential fluid velocity perturbation  $\beta$  that obeys an autonomous transport equation, plus a metric perturbation  $\Pi$  that obeys a wave equation with  $\beta$  as a source.  $\Pi$  is defined both in the interior and exterior of the star, and obeys the same equation in the interior and exterior. (In the exterior the matter source  $\beta$  simply vanishes.)

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$\beta$  requires no matching because the stellar surface is formed by fluid worldlines and is therefore a characteristic of the transport equation for  $\beta$ . The matching condition for  $\Pi$  is simply that it is continuous at the surface.

In Paper I we also characterized the  $l \geq 2$  *polar* perturbations by fields  $\chi$ ,  $k$  and  $\psi$  that admit free Cauchy data and evolve autonomously. By definition, these are all metric perturbations, but we showed that while  $\chi$  characterizes gravitational waves,  $k$  characterizes the sound waves, and  $\psi$  characterizes a second tangential fluid velocity perturbation. (The matter perturbations properly speaking can be reconstructed from these primary variables by quadratures if they are needed.) Consequently,  $\chi$  and  $k$  obey wave equations and  $\psi$  a transport equation (all of which are coupled). It is well known that the  $l \geq 2$  polar perturbations in the exterior are gravitational waves characterized by a single field  $Z$  that obeys a wave equation (the Zerilli equation [5]).  $\psi$  is transported along the fluid, and therefore, like the perturbation variable  $\beta$ , requires no explicit matching. The variables  $\chi$ ,  $k$  in the interior, and  $Z$  in the exterior, however, obey wave equations. This means that these three variables can and must be specified on a timelike boundary. The matching problem for the polar perturbations therefore consists in finding  $\chi$  and  $k$  just inside the surface from  $Z$  and its derivatives just outside (injection), and finding  $Z$  just outside from  $\chi$ ,  $k$ ,  $\psi$  and their derivatives just inside (extraction).

Matching the polar perturbations in the special cases  $l = 0$  and  $l = 1$ , which do not admit (completely) gauge-invariant perturbations and therefore require (partial) gauge-fixing, is also discussed.

In the appendixes we compare our results to previous work, and give some intermediate steps of the calculation.

## II. JUNCTION CONDITIONS ON A HYPERSURFACE

### A. Choice of continuous perturbation objects

Given a spacetime  $(M^4, g_{\mu\nu})$  containing matter, we want to derive junction conditions on a hypersurface  $\Sigma$  where the stress-energy tensor is allowed to be discontinuous. Our principal application will be the surface of a star, with fluid matter in the interior, and vacuum in the exterior, but for now we still allow the matching surface to be either spacelike or timelike. We assume that the metric is at least twice differentiable on each side of the boundary, but we do not assume that one coordinate patch covers both sides of the boundary. Therefore we do not assume that the metric components or their derivatives are continuous.

The condition that the stress-energy does not have a  $\delta$ -function singularity on the surface (a “surface layer”) translates into the condition that the induced metric and extrinsic curvature of the surface are the same on both sides of the surface [6]. These tensors are

$$i_{\mu\nu} \equiv g_{\mu\nu} \mp n_\mu n_\nu, \quad (1)$$

$$e_{\mu\nu} \equiv n_{\mu;\alpha} i^\alpha{}_\nu = n_{\mu;\nu} \mp n_{\mu;\alpha} n^\alpha{}_\nu, \quad (2)$$

where  $n^\mu$  is the unit vector orthogonal to the surface, with  $n_\mu n^\mu = \pm 1$ . Here and throughout this paper the upper (lower) sign applies when  $n^\mu$  is spacelike (timelike). (Therefore the upper sign will apply when we restrict to a stellar surface later.) The semicolon denotes a covariant derivative with respect to the metric  $g_{\mu\nu}$ . The tensors  $e_{\mu\nu}$  and  $i_{\mu\nu}$  are symmetric, and are intrinsic to the hypersurface in the sense that their contraction with  $n^\mu$  vanishes.

It is helpful to think of the two sides of the surface as two distinct spacetimes, each with a boundary. The two halves match without a surface layer if the 3-tensors  $e_{\mu\nu}$  and  $i_{\mu\nu}$  are equal. This in turn means that one can locally find 3 coordinates on each boundary so that the components of  $e_{\mu\nu}$  and  $i_{\mu\nu}$  are equal. To ask, for example, if  $n^\mu$  is continuous is not meaningful.

Clearly, the matching conditions for linear perturbations of the spacetime will be the continuity of the linear perturbations of  $e^{\mu\nu}$  and  $i^{\mu\nu}$ , but there is a complication: because the metric itself is perturbed, the perturbations of a tensor with its indices in different positions are no longer the same. For the metric itself, for example, with the notation

$$\Delta(g_{\mu\nu}) \equiv h_{\mu\nu}, \quad (3)$$

and the convention that all indices are moved with the background metric  $g_{\mu\nu}$  and its inverse  $g^{\mu\nu}$ , one has

$$\Delta(g^{\mu\nu}) = -h^{\mu\nu}. \quad (4)$$

Similarly, for the perturbation of any tensor  $t_{\mu\nu}$ ,

$$\Delta(t_{\mu\nu}) = g_{\mu\rho} g_{\nu\lambda} \Delta(t^{\rho\lambda}) + h_{\mu\lambda} t^\lambda{}_\nu + t_\mu{}^\lambda h_{\lambda\nu}. \quad (5)$$

Which of  $\Delta(e_{\mu\nu})$  and  $\Delta(e^{\mu\nu})$ , and of  $\Delta(i_{\mu\nu})$  and  $\Delta(i^{\mu\nu})$ , should be made continuous? We are not aware of an argument in the literature, and therefore give one here.

Let the surface be defined by the level surface  $f = 0$  of a scalar field  $f$ . The unit normal vector is then the normalized version of the gradient  $f_{,\mu}$ ,

$$\varphi \equiv (\pm f_{,\lambda} f_{,\rho} g^{\lambda\rho})^{-1/2}, \quad n_\mu \equiv \varphi f_{,\mu}, \quad n^\mu \equiv g^{\mu\nu} n_\nu. \quad (6)$$

The tangent vector to any curve in the surface  $f = 0$  gives us a vector field  $X^\mu$  that is by definition intrinsic to the surface. The gradient field  $f_{,\mu}$  is normal to any of these vectors in the sense that  $X^\mu f_{,\mu} = 0$ , and this fact is independent of the metric. Similarly, a contravariant tensor  $t^{\mu\nu\dots}$  is intrinsic, independently of the metric, if and only if  $t^{\mu\nu\dots} f_{,\mu} = 0$  on any of its indices. By extension we may define  $t_{\mu\nu\dots} = g_{\mu\alpha} g_{\nu\beta} \dots t^{\alpha\beta\dots}$  to be intrinsic, but this definition of an intrinsic covariant tensor depends on the metric. Therefore, it is clear that the perturbations  $\Delta(i^{\mu\nu})$  and  $\Delta(e^{\mu\nu})$  must be continuous (and so  $\Delta(i_{\mu\nu})$  and  $\Delta(e_{\mu\nu})$  are not).

In defining linear perturbations of a spacetime by subtracting fields at a point in the perturbed spacetime from the corresponding fields at a point in the unperturbed spacetime, the well-known problem arises that there is no unique or preferred map that identifies points in the unperturbed and perturbed spacetime. Changing the map (infinitesimally) changes the (linear) perturbations, even if the gauge has been fixed in the background. The point identification map is in practice provided by introducing coordinates on the perturbed and unperturbed spacetimes, and the linear gauge freedom in the perturbations then arises as the absence of a unique or preferred coordinate system on the perturbed spacetime, even if the background has a preferred coordinate system adapted to its symmetries. The gauge freedom in the linear perturbations arises over and above the gauge freedom in the background.

The surface of a star has a coordinate-independent significance, and it is therefore natural to identify the perturbed with the unperturbed surface. In fact, subtracting a background spacetime from the perturbed spacetime makes sense *only* if the point identification map maps the perturbed surface to the unperturbed surfaces: otherwise we would have to subtract, for example, a point in the fluid interior of the perturbed star from a point in the vacuum exterior of the background star. The resulting density “perturbation” would then be the background density, which in general is not small on the surface.

To avoid this problem, we choose a perturbation gauge in which the perturbed surface coincides with the background surface, obtain the matching conditions, and then transform back to the perturbation gauge in which we actually want to work. To formalize this, let us call  $\Delta X$  the perturbation of a tensor  $X$  in a completely arbitrary gauge. By Lie-dragging  $\Delta X$  along the vector field

$$\xi^\mu \equiv \pm \varphi \Delta f n^\mu. \quad (7)$$

we obtain the same perturbation in a new, auxiliary, gauge:

$$\bar{\Delta} X \equiv \Delta X - \mathcal{L}_\xi X \quad (8)$$

In this gauge the surface has not moved, because we see that  $\bar{\Delta} f = 0$ . We call this the surface gauge. (This gauge is of course not unique. Rather out of the four gauge degrees of freedom in a completely arbitrary gauge, the three parallel to the surface remain free.) In the following, we carry out some intermediate steps in surface gauge, and indicate this by using an overbar. As long as our final result contains only gauge-invariant fields, it does not matter in which (partial) gauge we have obtained it.

## B. Continuous perturbation objects in terms of metric perturbations

In surface gauge, with  $\bar{\Delta} f = 0$ , the perturbation of the normal vector  $n_\mu$  is

$$\bar{\Delta}(n_\mu) = \pm \frac{1}{2} n^\lambda n^\rho \bar{h}_{\lambda\rho} n_\mu, \quad (9)$$

so that

$$\bar{\Delta}(n^\mu) = \pm \frac{1}{2} n^\lambda n^\rho \bar{h}_{\lambda\rho} n^\mu - \bar{h}^{\mu\nu} n_\nu. \quad (10)$$

Note that  $\bar{\Delta}(n_\mu)$  is proportional to  $n_\mu$ , while  $\bar{\Delta}(n^\mu)$  is not in general proportional to  $n^\mu$ . This is another reflection of the fact that the natural intrinsic objects are contravariant: if their contraction with  $n_\mu$  vanishes, so does their contraction with  $n_\mu + \bar{\Delta}(n_\mu)$ . For the perturbation of a covariant derivative we use the formula

$$\Delta(X^\mu{}_{;\nu}) = [\Delta(X^\mu)]_{;\nu} + \Delta(\Gamma^\mu{}_{\lambda\nu})X^\lambda \quad (11)$$

and its obvious extensions to other tensors, where the “perturbation of the Christoffel symbol” is a tensor:

$$\Delta\Gamma^\mu{}_{\lambda\nu} \equiv \frac{1}{2}g^{\mu\rho}(h_{\rho\lambda;\nu} + h_{\rho\nu;\lambda} - h_{\lambda\nu;\rho}). \quad (12)$$

The continuous perturbations in surface gauge are

$$\bar{\Delta}(i^{\mu\nu}) = -i^{\mu\alpha}i^{\nu\beta}\bar{h}_{\alpha\beta}, \quad (13)$$

$$\bar{\Delta}(e^{\mu\nu}) = \left[ \pm \frac{1}{2}e^{\mu\nu}n^\alpha n^\beta - (i^{\mu\alpha}e^{\nu\beta} + i^{\nu\beta}e^{\mu\alpha}) \right] \bar{h}_{\alpha\beta} - i^{\mu\alpha}i^{\nu\beta}n^\lambda \frac{1}{2}(\bar{h}_{\lambda\alpha;\beta} + \bar{h}_{\lambda\beta;\alpha} - \bar{h}_{\alpha\beta;\lambda}). \quad (14)$$

We can replace continuity of  $\bar{\Delta}(e^{\mu\nu})$  by continuity of the shorter expression

$$\bar{\Delta}(e^{\mu\nu}) - e_\alpha{}^\mu \bar{\Delta}(i^{\alpha\nu}) - e_\alpha{}^\nu \bar{\Delta}(i^{\mu\alpha}) = \pm \frac{1}{2}e^{\mu\nu}n^\alpha n^\beta \bar{h}_{\alpha\beta} - i^{\mu\alpha}i^{\nu\beta}n^\lambda \frac{1}{2}(\bar{h}_{\lambda\alpha;\beta} + \bar{h}_{\lambda\beta;\alpha} - \bar{h}_{\alpha\beta;\lambda}). \quad (15)$$

$\bar{\Delta}(i_{\mu\nu})$  and  $\bar{\Delta}(e_{\mu\nu})$  are not continuous, but one finds by explicit calculation that

$$i^{\mu\alpha}i^{\nu\beta}\bar{\Delta}(i_{\alpha\beta}), \quad i^{\mu\alpha}i^{\nu\beta}\bar{\Delta}(e_{\alpha\beta}) \quad (16)$$

are continuous. (Gerlach and Sengupta [3,4] take the continuity of these expressions, or rather their equivalent in a general gauge, as their starting point.)

### III. PERTURBATIONS OF SPHERICAL SYMMETRY

#### A. Spherical background

We now restrict ourselves to a spherically symmetric background, but still allow the matching surface to be either timelike (upper sign in all following equations) or spacelike (lower sign). As in Paper I, we write the spherical background metric in a 2+2 covariant decomposition as

$$g_{\mu\nu} = \text{diag}(g_{AB}, r^2\gamma_{ab}), \quad (17)$$

where  $\gamma_{ab}$  is the unit metric on the two-sphere, and the background stress energy tensor as

$$t_{\mu\nu} = \text{diag}(t_{AB}, Qr^2\gamma_{ab}). \quad (18)$$

Let  $n^A$  be the vector field normal to the matching surface, with length squared  $\pm 1$ . Let  $u^A$  be the vector field tangential to the surface, with length squared  $\mp 1$ . The 1+1 metric  $g_{AB}$  can be written in terms of this orthonormal basis as

$$g_{AB} = \pm(-u_A u_B + n_A n_B). \quad (19)$$

We also use the notation  $g_{AB|C} \equiv 0$  for the covariant derivative in two dimensions,  $v_A \equiv r^{-1}r_{,A}$ , and the frame unit derivatives  $\dot{f} \equiv u^A f_{,A}$  and  $f' \equiv n^A f_{,A}$ .

The induced metric and extrinsic curvature of the matching surface of the background spacetime are

$$i_{\mu\nu} = \text{diag}(\mp u_A u_B, r^2\gamma_{ab}), \quad (20)$$

$$e_{\mu\nu} = \text{diag}(\mp \nu u_A u_B, W r^2\gamma_{ab}). \quad (21)$$

Therefore the scalars  $r$ ,  $\nu \equiv n^A|_A$  and  $W \equiv n^A v_A = r'/r$  are continuous. Note that although  $\nu$  contains a derivative of  $n^A$ , it depends only on the surface itself. As  $r$  is continuous everywhere on the surface, its unit derivative  $\dot{r}$  along the surface, and therefore  $U \equiv u^A v_A$ , is continuous. From the continuity of  $\dot{r}$  and  $r'$  follows the continuity of  $r_{,A}r^{,A}$  and hence of the Hawking mass  $m$ . From the Einstein equations we have that  $n^A n^B t_{AB}$  and  $u^A n^B t_{AB}$  are continuous. The list of continuous quantities can be extended by taking dot-derivatives of continuous quantities. The quantities that are *not* required to be continuous include  $u^A u^B t_{AB}$ ,  $Q$ , the Gauss curvature  $\mathcal{R}$  of  $g_{AB}$ , and  $\mu \equiv u^A|_A$ . Note that at the surface of a perfect fluid star (where  $u_A$  coincides with the fluid 4-velocity)  $u^A n^B t_{AB}$  and  $Q$  vanish identically.

## B. Axial perturbations

In the following we use the notation of [1] for the gauge-dependent and the gauge-invariant matter and metric perturbations, without repeating all the definitions. That notation uses the general metric and stress-energy perturbations of Gerlach and Sengupta [2], adding only specific notation for perfect fluid matter. Our notation here is also compatible with the notation of Gerlach and Sengupta in their papers on the matching conditions [3,4]. Note that for the axial perturbations we do not need to use surface gauge, as  $\Delta f$  is a polar perturbation.

The general junction conditions for axial perturbations in the framework of Gerlach and Sengupta are continuity of

$$l \geq 1: \quad u^A h_A^{\text{axial}}, \quad (22)$$

$$l \geq 2: \quad h, \quad (23)$$

from continuity of the induced metric (13), and

$$l \geq 1: \quad u^A n^B \left( h_{A|B}^{\text{axial}} - h_{B|A}^{\text{axial}} + 2v_A h_B^{\text{axial}} \right), \quad (24)$$

$$l \geq 2: \quad n^A (h_A^{\text{axial}} - h_{,A}), \quad (25)$$

from continuity of the extrinsic curvature (15).

In the axial sector we have four gauge-dependent continuity conditions (for  $l \geq 2$ ) and one gauge freedom. Therefore we find three gauge-independent continuous objects:

$$l \geq 1: \quad \Pi \equiv \epsilon^{AB} (r^{-2} k_A)_{|B}, \quad (26)$$

$$l \geq 2: \quad n^A k_A, \quad u^A k_A. \quad (27)$$

$\Pi$  is of particular interest as it obeys a wave equation with source terms given purely by matter perturbations. The Einstein equations allow us to reconstruct  $k_A$  later from  $\Pi$  and the matter perturbation  $L_A$ , as

$$l \geq 1: \quad (l-1)(l+2)k_A = 16\pi r^2 L_A - \epsilon_{AB} (r^4 \Pi)^{|B}. \quad (28)$$

Here  $\epsilon_{AB}$  is the totally antisymmetric covariant tensor with respect to  $g_{AB}$ . It can be given in terms of the basis as

$$\epsilon_{AB} = \pm (n_A u_B - u_A n_B) \quad (29)$$

We can use this to translate the continuity conditions on  $k_A$  into conditions on  $\Pi$ , the only dynamical variable we work with. As  $\Pi$  is continuous at all times at the matching surface, its derivative along the unit vector  $u^A$  in the surface,  $\dot{\Pi}$ , must also be continuous. Multiplying (28) by  $n^A$  and using (27) when  $l \geq 2$ , we find continuity of

$$l \geq 1: \quad n^A L_A. \quad (30)$$

This condition must be obeyed automatically if we have chosen the matching surface consistently with the matter equations of motion. Multiplying (28) by  $u^A$  we find that

$$l \geq 1: \quad \Pi' - 16\pi r^{-2} u^A L_A \quad (31)$$

is continuous, where we have used continuity of  $r'$ . This is a condition that needs to be imposed on the initial data for  $\Pi$  and the matter. It is compatible with the matter conservation equation. The continuity of  $\Pi$  itself needs to be imposed during the time evolution as a dynamical boundary condition.

For  $l = 1$  the variable  $\Pi$  is still gauge-invariant, but it no longer obeys a wave equation. Instead it is determined by the matter perturbations [1] through

$$l = 1: \quad r^4 \Pi = 16\pi T, \quad \text{where} \quad r^2 L_A = \epsilon_{AB} T^{|B}. \quad (32)$$

The continuity of  $\Pi$  implies the continuity of  $T$ , and (30) and (31) still hold. For  $l = 0$ , there are no gauge-invariant axial perturbations. Our results coincide exactly with those of Gerlach and Sengupta in [3].

### C. Polar perturbations

The general junction conditions for polar perturbations in surface gauge are continuity of

$$l \geq 0 : \quad \bar{i}_1 = u^A u^B \bar{h}_{AB}, \quad (33)$$

$$\bar{i}_2 = \bar{K}, \quad (34)$$

$$l \geq 1 : \quad \bar{i}_3 = u^A \bar{h}_A^{\text{polar}}, \quad (35)$$

$$l \geq 2 : \quad \bar{i}_4 = \bar{G}. \quad (36)$$

from continuity of the induced metric, and

$$l \geq 0 : \quad \bar{e}_1 = u^A u^B n^C (\bar{h}_{CA|B} + \bar{h}_{CB|A} - \bar{h}_{AB|C}) + n^C {}_C n^A n^B \bar{h}_{AB}, \quad (37)$$

$$\bar{e}_2 = n^A \left( 2\bar{h}_{AB} v^B - \bar{K}_{,A} - 2v_A \bar{K} - \frac{l(l+1)}{r^2} h_A^{\text{polar}} \right) \mp n^C v_C n^A n^B \bar{h}_{AB}, \quad (38)$$

$$l \geq 1 : \quad \bar{e}_3 = u^A n^B (\bar{h}_{AB} - \bar{h}_{A|B}^{\text{polar}} + \bar{h}_{B|A}^{\text{polar}} - 2v_A \bar{h}_B^{\text{polar}}), \quad (39)$$

$$l \geq 2 : \quad \bar{e}_4 = n^A \left( \bar{h}_A^{\text{polar}} - \frac{1}{2} r^2 \bar{G}_{,A} - r^2 v_A \bar{G} \right) \quad (40)$$

from continuity of the extrinsic curvature. Note that these are all scalars, and from their continuity at all time follows that of their derivative along the surface, e.g. of  $u^A K_{,A} \equiv \dot{K}$ .

Now we go back to the general gauge in which  $\Delta f$  is arbitrary. The continuous fields  $i_1$  to  $e_4$  in general gauge are  $\bar{i}_1$  to  $\bar{e}_4$  plus terms proportional to  $\Delta f$  and its derivatives. Next we find those linear combinations that are gauge-invariant for  $l \geq 2$ . As a rule of thumb, this is done by eliminating  $G$  and both components of the vector  $p_A \equiv h_A^{\text{polar}} - (1/2)r^2 G_{,A}$ . From this argument we would expect “8 – 3 = 5” continuous gauge-invariant fields. But we note that

$$N \equiv \varphi \Delta f - p^A n_A = \varphi (\Delta f - p^A f_{,A}), \quad (41)$$

is the gauge-invariant perturbation of the scalar  $f$  times the background quantity  $\varphi$ . Therefore  $N$  is gauge-invariant. In surface gauge it reduces to  $-n^A p_A = -e_4$ , which is continuous. Therefore we have 6 rather than 5 continuous gauge-invariant perturbations. ( $N$  is the same variable as the  $N$  of Gerlach and Sengupta [4].)

$N$  describes the deformation of the surface: while it is formally gauge-invariant in the bulk, it has a physical meaning only on the surface. To understand its significance better, we note that the perturbed  $f$  vanishes on the perturbed surface, so that

$$[f + \Delta f](x^\mu + \Delta x^\mu) = \Delta f + f_{,\mu} \Delta x^\mu + O(\Delta^2) = 0, \quad (42)$$

so that, in any gauge,

$$N = -\varphi \Delta x^A f_{,A} - p^A n_A = -n_A (\Delta x^A + p^A). \quad (43)$$

The “surface displacement”  $\Delta x^\mu$  is a gauge-dependent perturbation that transforms as vector field. Regge-Wheeler (RW) gauge is by definition the gauge where  $p^A = 0$ , so we can characterize  $-N$  as the normal proper displacement of the surface in RW gauge. Note that this is a gauge-dependent statement. In comoving gauge, for example, the surface is by definition not displaced.

The six continuous fields are

$$l \geq 0 : \quad I_1 = u^A u^B k_{AB} + 2\nu N, \quad (44)$$

$$I_2 = k \mp 2WN, \quad (45)$$

$$E_1 = \mp 2\mathcal{R}N + \nu n^A n^B k_{AB} + u^A u^B n^C k_{AB|C} - 2 \left( u^A n^B k_{AB} - \dot{N} \right) \cdot - 2\nu^2 N, \quad (46)$$

$$E_2 = \pm W n^A n^B k_{AB} - k' \mp 8\pi u^A u^B t_{AB} N \pm \left( U^2 - 3W^2 + \frac{1 \pm l(l+1)}{r^2} \right) N \mp 2U \left( u^A n^B k_{AB} - \dot{N} \right), \quad (47)$$

$$l \geq 1 : \quad E_3 = u^A n^B k_{AB} - 2\dot{N} + 2UN, \quad (48)$$

$$l \geq 2 : \quad E_4 = N. \quad (49)$$

These fields are continuous in any gauge and for the values of  $l$  indicated. They are gauge-invariant for  $l \geq 2$ , but only partially gauge-invariant for  $l = 1$ , and not gauge-invariant for  $l = 0$ .

#### IV. PERFECT FLUID MATTER

We now specialize to perfect fluid matter, and to the case where  $n^A$  is spacelike. We therefore take the upper sign in the equations above. In the stress-energy tensor, we have

$$u^A u^B t_{AB} = \rho, \quad n^A n^B t_{AB} = p, \quad u^A n^B t_{AB} = 0, \quad Q = p. \quad (50)$$

The surface on the star is defined by  $p = 0$ , and  $p$  is therefore continuous, but  $\rho$  can be discontinuous. The fluid four-velocity is tangent to the surface, and therefore we use the notation  $u^A$  for both the unit tangent vector to the surface, and for the fluid four-velocity inside, and we use  $n^A$  both for the vector that is normal to the surface, and normal to the fluid four-velocity in the interior of the star. As in paper I, we choose  $n^A$  to point outside and  $u^A$  to the future.  $f$  must then increase with radius to obtain  $f' > 0$  in accordance with the definition (6).

##### A. Axial perturbations

In the interior, the dynamical degrees of freedom are a matter velocity perturbation  $\beta$  and a metric perturbation  $\Pi$ . In order to work with fields that are  $O(1)$  at the origin, we rescale them as

$$\beta \equiv r^{l+1} \bar{\beta}, \quad \Pi \equiv r^{l-2} \bar{\Pi}, \quad (51)$$

$\bar{\beta}$  obeys an autonomous transport equation (it is, of course, transported along with the background fluid), while  $\bar{\Pi}$  obeys a wave equation with  $\bar{\beta}$  as a source term (Eqs. (65) and (71) of Paper I). In the exterior, the matter perturbation  $\bar{\beta}$  is not defined, but the metric perturbation  $\bar{\Pi}$  is, and it obeys the same field equation just without its  $\bar{\beta}$  source term. We can therefore work with the same equations of motion in both the interior and exterior, with  $\bar{\beta}$  defined to vanish identically in the exterior.  $\bar{\beta}$  parameterizes tangential fluid motion, and therefore there is no reason why it should vanish just inside the surface. It will therefore be discontinuous at the stellar surface. This discontinuity is consistent with the equation of motion, as  $\bar{\beta}$  is transported along fluid worldlines, and so could be discontinuous between any two fluid world lines.

Continuity of  $\Pi$  and  $\dot{\Pi}$  is equivalent to continuity of  $\bar{\Pi}$  and  $\dot{\bar{\Pi}}$ , as  $r$  and  $\dot{r}$  are continuous. For perfect fluid matter, the stress-energy perturbation  $L_A$  is given by

$$L_A = \beta(\rho + p)u_A, \quad (52)$$

and so  $L_A n^A$  vanishes identically. It is therefore automatically continuous, as suggested above. Continuity of (31) is equivalent to continuity of

$$\bar{\Pi}' + 16\pi r \rho \bar{\beta} \quad (53)$$

in terms of the rescaled variables, where we have used continuity of  $\bar{\Pi}$  and  $r'$ .

Matching for the axial perturbations is straightforward. The continuity of (53) is a constraint on the initial data  $\bar{\Pi}$ ,  $\dot{\bar{\Pi}}$  and  $\bar{\beta}$ . It is conserved by the evolution equations. (Note that  $\bar{\Pi}'$  is generally discontinuous if  $\rho \neq 0$  on the surface.) During the evolution, one has to impose the continuity of  $\bar{\Pi}$ . The case  $l = 1$  requires no matching at all, as  $\Pi$  is not a dynamical variable.

##### B. Polar perturbations

The variable  $N$  is not defined in the bulk but only on the boundary. As we want matching conditions for the bulk variables, we must eliminate  $N$  from the matching conditions. Here we do this for perfect fluid matter.

The stellar surface is defined by  $p = 0$ , and so it is natural to identify  $f$  with  $-p$ , with the minus sign chosen so that  $f$  increases with radius. It does not matter that  $p$  is not defined in the exterior, as we shall only need to calculate  $N$  just inside the surface. So  $N$  is related to the pressure perturbation, which in turn is determined by the density perturbation  $\omega$  and entropy perturbation  $\sigma$  through the equation of state. In appendix C, we derive the following result:

$$l \geq 0 : \quad N = - \left. \frac{c_s^2 \omega + C \sigma}{\nu} \right|_{\text{just inside the surface}}. \quad (54)$$

Note that if  $c_s^2$  vanishes on the surface (as it does for example for polytropic equations of state)  $\omega$  diverges while  $N$  is finite.

A second approach to calculating  $N$  is to relate the time derivative of the radial surface displacement  $N$  to the radial fluid velocity perturbation  $\gamma$ . In appendix C, we show that this leads to the expression

$$l \geq 0 : \quad \dot{N} - \mu N = - \left( \gamma + \frac{\psi}{2} \right)_{\text{just inside the surface}}. \quad (55)$$

We can use this equation as a better behaved alternative to (54) by integrating it as an ODE in time along the surface.

If one combines the dot-derivative of (54) with (55), one does not obtain an identity, but a new boundary condition. Using the equation for  $\dot{\sigma}$ , and the fact that  $c_s^2$  and  $C$  are constant on the surface, it can be written as

$$l \geq 0 : \quad c_s^2 \dot{\omega} = \left( \mu + \frac{\dot{\nu}}{\nu} \right) (c_s^2 \omega + C\sigma) + \left( \gamma + \frac{\psi}{2} \right) (\nu + Cs') \quad \text{on the surface.} \quad (56)$$

This equation relates  $\dot{\omega}$  to  $\gamma$ , while the matter equation of motion in the bulk relates  $\dot{\omega}$  to  $\gamma'$  (Eq. (99) of Paper I). Using the bulk equation for  $\dot{\omega}$ , we obtain a relation between  $\gamma$  and  $\gamma'$ . This new equation can be used as a dynamical boundary condition governing the reflection of sound waves at the boundary. But its origin is the purely kinematic boundary condition that the Lagrangian pressure perturbation  $\Delta p$  vanishes at the surface.

We can now address the continuity conditions for the bulk variables. We use the Einstein equations to replace  $\mathcal{R}$ . We also express the tensor  $k_{AB}$  in terms of its components in the fluid frame. Finally, we assume that the stellar surface is timelike, and so pick the upper sign. The continuous perturbation fields then become

$$l \geq 0 : \quad I_1 = \chi + k - 2\eta + 2\nu N, \quad (57)$$

$$I_2 = k - 2WN, \quad (58)$$

$$E_1 = \left[ 8\pi(\rho + p) - \frac{4m}{r^3} \right] N + \nu(\chi + k) + (\chi + k - 2\eta)' + 2\mu\psi + 2\dot{\psi} + 2\dot{N} - 2\nu^2 N, \quad (59)$$

$$E_2 = W(\chi + k) - k' - 8\pi\rho N + \left( U^2 - 3W^2 + \frac{1 \pm l(l+1)}{r^2} \right) N + 2U(\psi + \dot{N}), \quad (60)$$

$$l \geq 1 : \quad E_3 = -\psi - 2\dot{N} + 2UN, \quad (61)$$

$$l \geq 2 : \quad E_4 = N, \quad (62)$$

where  $N$  is given in terms of the matter perturbations by either (54) or (55). Note again that while these fields are continuous in any gauge, for the values of  $l$  indicated, they are completely gauge-invariant only for  $l \geq 2$ . We now consider the three cases  $l = 0$ ,  $l = 1$  and  $l \geq 2$  separately.

### 1. $l \geq 2$

For perfect fluid matter,  $\eta = 0$  for  $l \geq 2$  by virtue of one of the perturbed Einstein equations. Taking linear combinations of the six continuous fields  $I_1$  to  $E_4$  and their dot-derivatives, we obtain a simplified set of continuous gauge-invariant fields:

$$l \geq 2 : \quad C_1 = N, \quad (63)$$

$$C_2 = k, \quad (64)$$

$$C_3 = \chi, \quad (65)$$

$$C_4 = \psi, \quad (66)$$

$$C_5 = k' + 8\pi\rho N, \quad (67)$$

$$C_6 = \chi' + 2\mu\psi. \quad (68)$$

Note that the coefficients  $\rho$  and  $\mu$  can be discontinuous on the stellar surface. Note also that we do not need the value of  $N$  in the exterior, as its coefficient  $8\pi\rho$  vanishes there.

## 2. $l = 1$

For  $l = 1$ ,  $\eta$  does not vanish, and  $E_4$  is no longer continuous. As our variables are now only partially gauge-invariant, we make the partial gauge choice  $k = 0$ . In this partial gauge, the following five fields are continuous, and gauge-invariant under the remaining gauge freedom:

$$l = 1, k = 0: \quad \tilde{C}_1 = N, \quad (69)$$

$$\tilde{C}_2 = \chi - 2\eta, \quad (70)$$

$$\tilde{C}_3 = \psi, \quad (71)$$

$$\tilde{C}_4 = W\chi - 8\pi\rho N, \quad (72)$$

$$\tilde{C}_5 = (\nu + W)\chi + (\chi - 2\eta)' + 2\mu\psi. \quad (73)$$

Note that  $\chi$  is not continuous for  $l = 1$ .

## 3. $l = 0$

For  $l = 0$ , both  $E_3$  and  $E_4$  are no longer continuous. Our variables are now not gauge-invariant at all. We begin by fixing the gauge partially, again by setting  $k = 0$ . The four continuous (but gauge-dependent) fields in this partial gauge can be written as

$$l = 0, k = 0: \quad \hat{C}_1 = N, \quad (74)$$

$$\hat{C}_2 = \chi - 2\eta, \quad (75)$$

$$\hat{C}_3 = W\chi - 8\pi\rho N + 2U\psi, \quad (76)$$

$$\hat{C}_4 = (\nu + W)\chi + (\chi - 2\eta)' + 2(\mu + U)\psi + 2\dot{\psi}. \quad (77)$$

Note that  $\chi$  and  $\psi$  are not continuous for  $l = 0$ .

# V. MATCHING CONDITIONS FOR THE POLAR PERTURBATIONS

The physical  $l \geq 2$  polar perturbations comprise fluid convection, sound waves and gravitational waves in the interior, characterized by variables  $\psi$ ,  $k$  and  $\chi$  respectively, but all coupled together. In the exterior there are only gravitational waves, and these can be characterized by a single variable  $Z$  first found by Zerilli. The matching is non-trivial because at the surface  $Z$  does not simply coincide with  $\chi$ .

To achieve clarity, we formulate the matching conditions as two sets of boundary conditions.  $\chi$ ,  $\psi$  and  $k$  are considered as evolving on a spacetime with a timelike boundary, on which one can freely specify certain boundary conditions. The most natural ones are Dirichlet boundary conditions for  $\chi$  and  $k$ , the two variables that obey wave equations. Similarly,  $Z$  obeys a wave equation on another spacetime with timelike boundary, and one can freely specify a Dirichlet boundary condition for it. In this view the matching problem consists in finding Dirichlet boundary conditions for  $Z$  given  $\chi$ ,  $\psi$  and  $k$  and their first derivatives in the interior (extraction), and Dirichlet boundary conditions for  $\chi$  and  $k$  given  $Z$  and its first derivatives in the exterior (injection).

## A. Vacuum exterior

The exterior spacetime in spherical symmetry must be the Schwarzschild spacetime, with metric

$$ds^2 = -a^{-2} dt^2 + a^2 dr^2 + r^2 d\Omega^2, \quad (78)$$

where  $a^2$  is shorthand for  $(1 - 2m/r)^{-1}$ . Here  $r$  and  $t$  are the Schwarzschild coordinates, and  $m$  is the mass of the star.  $r$  is identical with the area radius  $r$ , and  $m$  is identical with the Hawking mass  $m$ , which is constant in the exterior. The matching conditions in the background spacetime are continuity of  $r$  and  $m$  at the stellar surface. As in Paper I, we use a hat to distinguish the radial frame from the fluid frame. On the Schwarzschild background, in the radial frame, we have

$$\hat{U} = \hat{\mu} = 0, \quad \hat{W} = (ar)^{-1}, \quad \hat{\nu} = (a^{-1})_{,r} = mr^{-2}a. \quad (79)$$

The frame derivative  $\hat{\cdot}$  of all these quantities also vanishes. The frame derivatives expressed in Schwarzschild coordinates are  $f^{\hat{j}} = a^{-1}f_{,r}$  and  $\hat{f} = af_{,t}$ . It is useful to introduce the tortoise radius  $r_*$  by  $dr_* = a^2(r)dr$ , so that  $f^{\hat{j}} = af_{,r_*}$ .

Zerilli [5] first found that the  $l \geq 2$  polar perturbations of Schwarzschild obey a single wave equation for a single variable, with all other perturbation variables obtained by quadratures. A gauge-invariant single variable obeying the same wave equation (the Zerilli equation) was constructed by Moncrief [7]. In appendix A we rederive the Moncrief variable and the Zerilli equation in our framework. The final result is that the variable

$$Z \equiv A(r)\hat{\chi} + B(r)k + C(r)k^{\hat{j}} \quad (80)$$

with

$$A = \frac{2r}{a^2(l^2 + l + 1) - 3}, \quad B = \frac{r[a^2(l^2 + l + 1) - 1]}{a^2(l^2 + l + 1) - 3}, \quad C = \frac{-2ar^2}{a^2(l^2 + l + 1) - 3}, \quad (81)$$

obeys the wave equation

$$Z^{|A|}_{|A} - V(r)Z = -\hat{\ddot{Z}} + Z^{\hat{\prime\prime}} + \hat{\nu}Z^{\hat{j}} - V(r)Z = 0, \quad (82)$$

with the potential

$$V(r) = \frac{l(l+1)}{r^2} - \frac{6m}{r^3} \frac{r^2\lambda(\lambda+2) + 3m(r-m)}{(r\lambda+3m)^2}, \quad (83)$$

where  $\lambda \equiv (l+2)(l-1)/2$ . Note that the Zerilli equation can also be written as  $-Z_{,tt} + Z_{,r_*r_*} - a^{-2}V(r)Z = 0$  using the tortoise coordinate. The initial data  $Z, \hat{\dot{Z}}$  can be set freely on a Cauchy surface, and  $Z$  evolves autonomously. The other metric perturbations are given algebraically in terms of derivatives of  $Z$ , and obey the Einstein equations automatically.

For  $l = 0$  and  $l = 1$  all exterior perturbations are pure gauge. We review this in appendix B.

## B. $l \geq 2$ matching

In the interior the principal part of the free evolution equations is  $-\ddot{\chi} + \chi'' + \dots\psi', -\ddot{k} + c_s^2k'' + \dots\psi'$  and  $\dot{\psi}$ . The evolution equations therefore have 5 characteristics, namely the light cone, the sound cone, and the fluid 4-velocity. Two of these characteristics travel from the outside to the inside, and so two quantities can be determined freely on the stellar surface. A convenient choice of these is  $\chi$  and  $k$ . Similarly, the principal part of the free evolution equation in the exterior is  $-\ddot{Z} + Z''$ , so that we can specify one quantity freely on the stellar surface. It is usefully chosen to be  $Z$ . The matching problem now consists in determining  $Z$  just outside from  $\chi, k$  and  $\psi$  and their derivatives just inside (extraction), and  $\chi$  and  $k$  just inside from  $Z$  and its derivatives just outside (injection).

In section IV, we gave the continuous quantities  $C_1$  to  $C_6$  in tensor components in the fluid frame. In order to write down equalities between quantities just inside and just outside the stellar surface, we must express these quantities in terms of  $Z$  and its derivatives. For numerical work one would probably want to use comoving coordinates in the star and Bondi or polar-radial coordinates outside. Therefore we use fluid frame derivatives inside, but radial frame derivatives outside.

Frame tensor components and related quantities in any two frames are related by hyperbolic rotations through an angle  $\xi$  (for vectors) or  $2\xi$  (for 2-tensors). We use the formulas given in Appendix E of Paper I, and take hatted quantities to refer to the radial frame and unhatted quantities to the fluid frame. From (79), and Eq. (E4) of Paper I, we find that

$$\sinh \xi = arU, \quad \cosh \xi = arW. \quad (84)$$

These expressions are evaluated just inside the surface.

We write down some matching conditions that we shall need as intermediate results. The continuous fields  $C_1 = N$  and  $C_2 = k$  are frame-independent, and therefore they are the same just inside and just outside the surface. The tensor components  $\phi \equiv \chi + k$  and  $\psi$  are also continuous, and we just need to transform then from the fluid to the radial frame. The continuity of  $C_3$  and  $C_4$  therefore gives

$$\phi = \cosh 2\xi \hat{\phi} - \sinh 2\xi \hat{\psi}, \quad (85)$$

$$\psi = \cosh 2\xi \hat{\psi} - \sinh 2\xi \hat{\phi}, \quad (86)$$

where the left-hand side is just inside, and the right-hand side just outside.  $k'$  and  $\dot{k}$  transform by a rotation through the angle  $\xi$ .  $\dot{k}$  is continuous because  $k$  is, but  $k'$  just outside is equal to  $k' + 8\pi\rho N$  just inside. Putting this together, the continuity of  $C_2$  and  $C_5$  gives

$$k' + 8\pi\rho N = \cosh \xi k' + \sinh \xi \dot{k}, \quad (87)$$

$$\dot{k} = \cosh \xi \dot{k} + \sinh \xi k'. \quad (88)$$

We obtain the extraction equation from the definition (80) of  $Z$ , the inverse of (85,86) and the inverse of (87,88). It is

$$Z = A (\cosh 2\xi \phi + \sinh 2\xi \psi) + (B - A)k + C [\cosh \xi (k' + 8\pi\rho N) - \sinh \xi \dot{k}], \quad (89)$$

where the left-hand side is evaluated just outside and the right-hand side just inside, and where  $A$ ,  $B$  and  $C$  are the coefficients defined in Eq. (81). Substituting these coefficients, the expressions (84), and the definition  $\phi = \chi + k$ , we obtain the final version of our extraction equation

$$Z = rk + \frac{2r^4}{(l+2)(l-1)r + 6m} [(W^2 + U^2)(\chi + k) + 2UW\psi - W(k' + 8\pi\rho N) + U\dot{k}]. \quad (90)$$

The injection equation for  $k$  is obtained from the constraint (A11) and continuity of  $k$ . The injection equation for  $\chi$  is obtained from the continuity equation (85), the definition  $\phi = \chi + k$ , the injection equation for  $k$ , and the constraints (A12) and (A13). The two injection equations are therefore

$$\begin{aligned} k &= (A11) \\ &= \frac{Z}{r} + \frac{2}{l(l+1)} \left(1 - \frac{2m}{r}\right) \left[ Z_{,r} - \frac{6m}{(l+2)(l-1)r + 6m} \frac{Z}{r} \right], \end{aligned} \quad (91)$$

$$\begin{aligned} \chi &= \cosh 2\xi (A12) - \sinh 2\xi (A13) + (\cosh 2\xi - 1) (A11) \\ &= \left(1 - \frac{2m}{r}\right)^{-1} r^2 [(W^2 + U^2) (A12) - 2UW (A13) + 2U^2 (A11)], \end{aligned} \quad (92)$$

where the equation numbers in brackets stand for the right-hand sides of those equations, evaluated just outside the surface, and where the left-hand sides are just inside the surface. We have not written out (92) in full because no important simplifications occur when one expands it. Equations (90-92) are the main results of this paper.

We have not used the continuity of  $C_6$ .

### C. $l = 1$

For  $l = 1$ , the polar perturbation variables we use are no longer fully gauge-invariant. It is necessary to make a partial gauge choice, and in Paper I we chose  $k = 0$ . This still leaves a residual gauge freedom worth one free function of time only. In paper I we fixed this remaining freedom by demanding that at the center the leading order in  $r$  of the variable  $\eta$  vanishes, or  $\bar{\eta} = O(r^2)$  in the notation of Paper I. In the exterior we also make the partial gauge choice  $k = 0$ , and again this leaves a residual gauge freedom parameterized by one free function of time. As reviewed in Appendix B, this freedom can be used to set *all* perturbation variables in the exterior equal to zero, which shows that the exterior perturbations are pure gauge. This gauge is not the same as the one fixed by imposing  $\bar{\eta} = O(r^2)$  at the center. We therefore drop the latter – it was imposed only in the absence of a better choice and does not simplify the field equations. The equations of Paper I are valid for any way of fixing the residual gauge freedom, and in particular for the gauge choice  $\eta = 0$  at the surface that we adopt now.

With the complete gauge choice  $k = 0$  everywhere and  $\eta = 0$  at the surface we find that  $\hat{\chi} = \hat{\psi} = \eta = 0$  in the exterior. Transforming this to the fluid frame, we have  $\chi = \psi = \eta = 0$  just outside the surface. From the continuity of  $\tilde{C}_2$ ,  $\tilde{C}_3$  and  $\tilde{C}_4$ , we have

$$l = 1, k = 0, \eta = 0 \text{ just outside : } \psi = 0, \quad (93)$$

$$\chi = 2\eta, \quad (94)$$

$$\eta = \frac{4\pi\rho}{W}N, \quad (95)$$

just inside the surface. These equations can be used as boundary conditions for the integration of the ODE system (A12-A14) of Paper I. From the continuity of  $W\hat{C}_5 - U\dot{\hat{C}}_2$ , we have

$$r|v|^2 D(\chi - 2\eta) + 8\pi\rho(\nu + W)N = 0, \quad (96)$$

just inside the surface. (See Paper I for the definition of the radial derivative  $D$ .) If we now use Eqs. (A13) and (A12) of paper I to eliminate  $D\eta$  and  $D\chi$ , (93-95) to eliminate  $\chi$ ,  $\psi$  and  $\eta$ , and (54) to eliminate  $N$ , and use  $p = 0$  at the surface, we obtain an identity.

For  $l = 0$  and  $l = 1$  we use the matter perturbations  $\omega$  and  $\gamma$  as dynamical variables. Together they describe sound waves, and therefore we need one boundary condition at the surface. It is given by (56).

#### D. $l = 0$

The situation for  $l = 0$  is similar to the one for  $l = 1$ . We have already made the partial gauge choice  $k = 0$ . In order to fix the gauge further, we impose the further gauge choice  $\hat{\psi} \equiv -\hat{n}^A \hat{u}^A k_{AB} = 0$ , where  $\hat{\psi}$  is the equivalent of the frame component  $\psi$  in the radial frame, instead of the fluid frame. (The reason for this choice is discussed in paper I.) This gauge choice is equivalent to

$$\hat{\psi} = 0, k = 0 : \quad \psi = \frac{2UW}{W^2 + U^2}(\eta - \chi). \quad (97)$$

We shall use this to eliminate  $\psi$  from  $\hat{C}_3$  and  $\hat{C}_4$ .

The gauge choice  $k = \hat{\psi} = 0$  still leaves a small residual gauge freedom. In Paper I we used this last gauge freedom to fix  $\bar{\eta} = O(r^2)$  at the center, but as for  $l = 1$  this is not the best choice to make in the presence of a vacuum exterior, where all perturbations are pure gauge. Instead we fix the gauge so that all perturbations vanish in the exterior (see Appendix B), so that  $k = \eta = \hat{\psi} = \hat{\chi} = 0$  everywhere outside, and therefore also  $\psi = \chi = 0$ . Continuity of  $\hat{C}_2$  and  $\hat{C}_3$  then gives rise to the boundary conditions

$$l = 0, k = 0, \hat{\psi} = 0, \eta = 0 \text{ just outside : } \chi = 2\eta, \quad (98)$$

$$\eta = \frac{W^2 + U^2}{W^2 - U^2} \frac{4\pi\rho}{W}N. \quad (99)$$

The continuity of  $\hat{C}_4$  gives again rise to an identity, similar to the  $l = 1$  case. Again, we need the matter boundary condition (56).

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### APPENDIX A: DERIVATION OF THE ZERILLI EQUATION

Here we only consider the case  $l \geq 2$ . Recall that in the interior,  $\chi$ ,  $\psi$  and  $k$  evolve autonomously and can be considered as the true degrees of freedom. Three constraints give the matter perturbations  $\alpha$ ,  $\gamma$  and  $\omega$  directly in terms of derivatives of  $\chi$ ,  $\psi$  and  $k$ . In vacuum, however, these matter perturbations vanish, so that the same equations become three constraints on the Cauchy data for  $\hat{\chi}$ ,  $\hat{\psi}$  and  $k$ . They are

$$0 = (\hat{k})^{\hat{}} - \hat{W}\hat{\chi} + \frac{l(l+1)}{2r^2}\hat{\psi}, \quad (\text{A1})$$

$$0 = -\hat{k}^{\hat{}} + \frac{l(l+1)}{r^2}(\hat{\chi} + k) - \frac{(l-1)(l+2)}{2r^2}\hat{\chi} + \hat{W}\hat{\chi}^{\hat{}} - 2\hat{W}k^{\hat{}}, \quad (\text{A2})$$

$$0 = \hat{\psi}^{\hat{}} + 2\hat{\nu}\hat{\psi} + \hat{\chi} + 2\hat{k}. \quad (\text{A3})$$

The evolution equations in vacuum are

$$-\hat{\chi} + \hat{\chi}^{\hat{}} = -2 \left[ 2\hat{\nu}^2 - \frac{6m}{r^3} \right] (\hat{\chi} + k) - (5\hat{\nu} - 2\hat{W})\hat{\chi}^{\hat{}} + \frac{(l-1)(l+2)}{r^2}\hat{\chi}, \quad (\text{A4})$$

$$-\hat{k} = -\hat{W}\hat{\chi}^{\hat{}} - \hat{\nu}k^{\hat{}} - \frac{4m}{r^3}(\hat{\chi} + k) - \frac{(l-1)(l+2)}{2r^2}\hat{\chi}, \quad (\text{A5})$$

$$-\hat{\psi} = 2\hat{\nu}(\hat{\chi} + k) + \hat{\chi}^{\hat{}} \quad (\text{A6})$$

Note that in obtaining the vacuum evolution equations from the evolution equations inside fluid matter, we have set  $c_s^2 = 0$ . Setting  $c_s^2$  to a different formal value would correspond to adding that value times the constraint (A2) to the evolution equation (A5). Here we have chosen to write the evolution equation for  $k$  in the form (A5) which does not contain  $k^{\hat{}}$ .

We note that in vacuum  $\hat{\psi}$  already plays a passive role, while  $\hat{\chi}$  already almost obeys an autonomous wave equation, containing only  $k$  as an unwanted term. This suggests looking for a variable of the form (80) that obeys a wave equation of the form (82) above, to be constructed from equations (A2), (A4) and (A5). Note that the new variable  $Z$  should not contain  $\hat{k}$  in order to avoid bringing  $\hat{\psi}$  back into the game.

Introducing the ansatz (80) in equation (82) and using the above equations to eliminate second derivatives, we obtain

$$-\hat{Z} + Z^{\hat{}} + \hat{\nu}Z^{\hat{}} - V_0\hat{\chi}^{\hat{}} - V_1\hat{\chi} - V_2k - V_3k^{\hat{}} = 0, \quad (\text{A7})$$

where the  $V_i$  are certain combinations of  $A, B, C$  and  $a$ . It is clear that  $V_0$  should vanish:

$$V_0 = \frac{2(aC + ra^2A)^{\hat{}}}{ra^2} = 0, \quad \Rightarrow \quad C = -raA + \frac{c_1}{a}, \quad (\text{A8})$$

where  $c_1$  is a constant. The coefficients  $V_1/A$ ,  $V_2/B$  and  $V_3/C$  have to be equal to each other and are all equal to the “scattering potential”  $V$ . Using our result for  $C$  this gives two coupled nonlinear differential equations of second order for  $A$  and  $B$ . With the choice  $c_1 = 0$  one of them is first order and can be easily integrated (define  $\Lambda_n = -n + a^2(l^2 + l + 1)$  for integer  $n$ ):

$$\frac{V_1}{A} - \frac{V_3}{C} = \frac{1}{raA} [-\Lambda_1 A + 2B]^{\hat{}} = 0, \quad \Rightarrow \quad B = \frac{\Lambda_1}{2}A + c_2, \quad (\text{A9})$$

where  $c_2$  is another constant. Having expressed  $B$  and  $C$  in terms of  $A$ , the other equation gives a single differential equation for  $A$ , which again can be easily integrated if  $c_2 = 0$ :

$$\frac{V_2}{B} - \frac{V_1}{A} = \frac{l(l+1)}{\Lambda_1} \frac{a}{A} \left( \Lambda_3 \frac{A}{r} \right)^{\hat{}} = 0, \quad \Rightarrow \quad A = \frac{rc_3}{\Lambda_3}. \quad (\text{A10})$$

We fix the overall constant  $c_3$  in the definition of  $Z$  as  $c_3 = 2$ , which makes our  $Z$  equal to the variable found by Moncrief. Using these expressions for  $A, B, C$  we finally obtain the coefficients given above in (81).

The initial data  $Z, \hat{Z}$  can be set freely on a Cauchy surface, and  $Z$  evolves autonomously. The metric perturbations  $k, \hat{\chi}$  and  $\hat{\psi}$  can be reconstructed from  $Z$  using the vacuum perturbative Einstein equations (A1), (A2) and (A3) given in the appendix. When we introduce the expression for  $Z$  into these equations the derivative terms cancel out and  $k, \hat{\chi}$  and  $\hat{\psi}$  are obtained as algebraic expressions in  $Z$  and its derivatives:

$$k = \frac{Z}{r} + \frac{2}{al(l+1)} \left( \frac{6a^{\hat{}}Z}{a\Lambda_3} + Z^{\hat{}} \right), \quad (\text{A11})$$

$$\hat{\chi} = -\frac{2Z}{r} - \frac{6a^{\hat{}}Z}{\Lambda_3} + \frac{2r}{l(l+1)} \left( \frac{6a^{\hat{}}Z}{a\Lambda_3} + Z^{\hat{}} \right)^{\hat{}}, \quad (\text{A12})$$

$$\hat{\psi} = -\frac{2}{l(l+1)} \left( \frac{a(l-1)(l+2)\hat{Z}}{\Lambda_3} + r(\hat{Z})^{\hat{}} \right). \quad (\text{A13})$$

Note that  $a' = -a^2 m/r^2$ . The remaining perturbative vacuum Einstein equations (A4), (A5) and (A6) are now linear combinations of the Zerilli equation and its derivatives.  $Z$  is therefore the true degree of freedom in the exterior.

## APPENDIX B: $L = 0, 1$ EXTERIOR POLAR PERTURBATIONS

a.  $l = 1$ :

Together with  $\hat{\chi}$ ,  $k$  and  $\hat{\psi}$ , now we also have the variable  $\eta$ . These variables are still invariant under gauge transformations generated by a vector field  $\xi^A$  on  $M^2$ , but not under gauge changes generated by  $r^2 \xi Y^{;a}$  with  $\xi$  a scalar field. We impose the condition  $k = 0$  to eliminate this freedom. The equations of motion are then greatly simplified:

$$\hat{\chi}' = -\frac{2a}{r}\hat{\chi}, \quad \hat{\dot{\chi}} = \frac{a}{r}\hat{\psi}, \quad (B1)$$

$$\hat{\psi}' = \frac{1-2a^2}{ar}\hat{\psi}, \quad \hat{\dot{\psi}} = \frac{a^2-1}{ar}(\hat{\chi} - 3\eta), \quad (B2)$$

$$\eta' = -\frac{\hat{\chi} + (-3 + 2a^2)\eta}{ar}. \quad (B3)$$

They can be integrated to give

$$\hat{\chi} = \frac{g(t)}{(r-2m)^2}, \quad \hat{\psi} = \frac{r\partial_t g(t)}{(r-2m)^2}, \quad \eta = \frac{2mg(t) - r^3\partial_t^2 g(t)}{6m(r-2m)^2}, \quad (B4)$$

where the free function of time  $g(t)$  is a residual gauge freedom, not eliminated by the condition  $k = 0$ . We are still allowed to perform changes of the form  $\xi(t, r) = a^2(r)f(t)$ , where  $f(t)$  is an arbitrary function of time. Under that change we find  $\Delta g(t) = 6mf(t)$ . We can use this freedom to choose  $g(t) = 0$ , so that all our perturbations vanish. This can be interpreted as a small displacement of the perturbed star so that the center of mass position is not perturbed, and therefore the vacuum exterior is not affected by the perturbation.

b.  $l = 0$ :

In this case we use the gauge  $k = \hat{\psi} = 0$ . The equations are even simpler:

$$\hat{\chi}' = -\frac{a}{r}\hat{\chi}, \quad \hat{\dot{\chi}} = 0, \quad \eta' = 0. \quad (B5)$$

Their solution is

$$\hat{\chi} = \frac{2\Delta m}{r-2m}, \quad \eta = g(t), \quad (B6)$$

where the constant  $\Delta m$  is a perturbation of the mass of the star and  $g(t)$  is an arbitrary function of time which again represents residual gauge freedom, now related to time reparameterization. To first order in perturbations, we can write the perturbed metric as

$$-\left(1 - \frac{2m + 2\Delta m}{r}\right) \{[1 + g(t)]dt\}^2 + \left(1 - \frac{2m + 2\Delta m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (B7)$$

The mass perturbation  $\Delta m$  is constant in time. In physical terms this is so because the star can lose or gain mass-energy only to second order in perturbation theory. We consider only  $\Delta m = 0$  here, and treat any change in the mass of the star as a change in the background solution.

## APPENDIX C: CALCULATION OF $N$ IN TERMS OF FLUID VARIABLES

We have derived Eq. (54) in two ways.

1. The surface of a perfect fluid star is defined by the vanishing of the pressure  $p$ . We identify  $f$  inside the star with the negative fluid pressure  $-p$  (the sign is chosen so that  $f' > 0$ ), so that  $N$  becomes

$$N = -(p^A p_{,A})^{-1/2} (\Delta p - p^A p_{,A}) = (p')^{-1} \rho (c_s^2 \omega + C\sigma) \quad \text{just inside the surface.} \quad (C1)$$

(Do not confuse the pressure gradient  $p_{,A}$  with the metric perturbation  $p_A$ . Note also that  $p^A p_{,A} = p'^2$  because  $p = 0$  on the surface at all times, and that  $p' < 0$ , which explains the overall sign in the second equality above.) From Eq. (44) of paper I we find that

$$p' = -\nu(\rho + p) \quad (C2)$$

and so we obtain Eq. (54) above. This derivation holds for all  $l$ .

2. We can also obtain (54) without identifying  $f$  explicitly with  $-p$ . The evolution equation for  $k$ , Eq. (88) of Paper I, holds on both sides of the surface if one formally sets  $\rho = 0$  in the exterior. It is therefore continuous. Let us first assume  $l \geq 2$ . We bring known continuous quantities such as  $\ddot{k}$  to one side, and find that

$$c_s^2(-k'' + \text{other metric perturbations}) - \nu k' + 8\pi\rho C\sigma = \text{continuous.} \quad (C3)$$

Now we use Eq. (94) of paper I to replace the entire term in round brackets by  $8\pi\rho\omega$ , and we use the continuity of  $C_5$  to replace  $k'$  by  $-8\pi\rho N$  plus a continuous term. (It is this step that brings in  $N$ .) We obtain

$$8\pi\rho(c_s^2\omega + C\sigma + \nu N) = \text{continuous} \quad (C4)$$

The left-hand side vanishes identically in the exterior because  $\rho = 0$ , so by continuity it must vanish just inside the surface as well, and we obtain Eq. (54). For  $l = 1$ , Eqs. (88) and (94) of Paper I still holds. We now have  $\eta \neq 0$  and, by gauge choice,  $k = 0$ , but the final result is the same. In the case  $l = 0$ , we can use Eqs. (A18) and (A19) instead of Eq. (88) to obtain once again the same result.

We have derived equation (55) for  $\dot{N}$  in two different ways, too.

1. The velocity perturbation is the Lie-derivative of the position perturbation along fluid world lines:

$$\Delta u^\mu = \mathcal{L}_u \Delta x^\mu \quad (C5)$$

and so we have, in any gauge, that

$$n_A \Delta(u^A) = n_A (u^B \Delta x^A_{|B} - \Delta x^B u^A_{|B}) = u^B (n_A \Delta x^A)_{|B} - u^B n_{A|B} \Delta x^A + \Delta x^B n_{A|B} u^A = (n_A \Delta x^A)' - \mu(n_A \Delta x^A), \quad (C6)$$

where we have used Eq. (42) of Paper I to eliminate  $n_{A|B}$ . A simple calculation using Eqs. (52), (55), (21) and (42) of Paper I gives

$$n_A \Delta(u^A) = \gamma + \frac{\psi}{2} - (n_A p^A)' + \mu(n_A p^A). \quad (C7)$$

Eliminating  $n_A \Delta u^A$  between the last two results, and using (43) gives us (55). This derivation holds for all  $l$ .

2. Alternatively, we can use the field equations directly. Assume  $l \geq 2$  for now. Using equation (93) of Paper I, extracting the continuous terms and using the continuity of  $C_5$  and  $\dot{C}_5$ , we find that

$$8\pi(\rho + p) \left( \dot{N} - \mu N + \gamma + \frac{\psi}{2} \right) = \text{continuous.} \quad (C8)$$

We use the same argument as above: this vanishes in the exterior, so it must vanish just inside the surface, which gives us (55). For  $l = 1$  and  $l = 0$  the derivation is similar, taking into account  $\eta \neq 0$  and  $k = 0$ .

## APPENDIX D: PERTURBATIONS OF A TIME-DEPENDENT STAR: COMPARISON WITH SEIDEL

In [8] Seidel considers axial perturbations of a general time-dependent spherically symmetric star. His variable  $U$  is  $-\beta$  in our notation. Our variable  $\Pi$  is related to the interior and exterior variables of Seidel as

$$\Pi \text{ (our notation)} = -\frac{\psi}{l(l+1)R^2} \text{ (Seidel interior)} = -\frac{\tilde{\psi}}{l(l+1)r^3} \text{ (Seidel exterior)}. \quad (\text{D1})$$

The continuity conditions of Seidel agree with the continuity of  $\Pi$  that we find.

In [9] Seidel derives evolution and matching equations for the  $l = 2$  gauge-invariant polar perturbations. He uses comoving coordinates in the interior and Schwarzschild coordinates in the exterior. His exterior perturbation variables  $\tilde{Q}_1$  and  $\tilde{\psi}$  are related to our variable  $Z$ , restricted to  $l = 2$ , by

$$\tilde{Q}_1 = (1 + \frac{3m}{2r})4Z, \quad \tilde{\psi} = \sqrt{\frac{4\pi}{5}} 4Z. \quad (\text{D2})$$

$\tilde{\psi}$  obeys the Zerilli equation. His variable  $\tilde{k}_1$  is equal to our  $k$ . We have already compared his interior variables with ours in Appendix C of Paper I.

Seidel notes that his interior variable  $q_4$  is gauge-invariant when restricted to the boundary and in any gauge where the perturbed boundary is at the same coordinate location as the unperturbed boundary (what we call surface gauge). He derives the matching conditions in that gauge. We have

$$A^{-1}q_4 = 2n^A p_A, \quad (\text{D3})$$

with Seidel's notation on the left and ours on the right. This is of course equal to  $-2N$  in the surface gauge  $\Delta f = 0$  in which Seidel works, and is therefore actually gauge-invariant, although Seidel does not make that point. Seidel gives an evolution equation for  $q_4$ , Equation (2.89), that can be cast in covariant form in our notation if we assume that the energy density ( $\rho$  in our notation,  $\eta \equiv (1 + \epsilon)\rho$  in the notation of Seidel) reduces to the rest mass density  $\rho$  of Seidel when  $p = 0$ . The rest mass density  $\rho$  of Seidel is linked to his background metric coefficient  $A$  through the gauge condition  $A^{-1} = 4\pi r^2 \rho$ . His equation (2.89) is then exactly our equation (55).

Seidel's matching equation (2.84), giving  $\tilde{Q}_1$  in terms of interior variables is equivalent to our extraction equation, Equation (90). His second matching equation (2.86), which gives  $k$  in terms of his exterior variable  $\tilde{Q}_1$  is equivalent to our Equation (A11). His equation (2.85) for  $\tilde{Q}_{1,r}$  is a linear combination of the first two.

## APPENDIX E: PERTURBATIONS OF OPPENHEIMER-SNYDER: COMPARISON WITH CPM

Cunningham, Price and Moncrief study the axial [10] (CPM1) and polar [11] (CPM2) perturbations of spherically symmetric homogeneous dust collapse (Oppenheimer-Snyder collapse). The background solution consists of a spherical segment of a dust-filled closed Friedmann solution in its collapsing phase matched to a Schwarzschild exterior. The interior metric, in the notation of CPM, is

$$ds^2 = -d\tau^2 + R^2(\tau) (d\chi^2 + \sin^2 \chi d\Omega^2). \quad (\text{E1})$$

Fluid elements are at constant  $\chi$ . The stellar surface is at  $\chi = \chi_0$ , corresponding to  $r = r_0(\tau) = \sin \chi_0 R(\tau)$ . These are just a special case of comoving coordinates, and from Appendix C of [1] we can read off that in this metric, in a fluid frame,

$$\mu = U = \frac{1}{R(\tau)} \frac{dR}{d\tau}, \quad \mu' = U' = 0, \quad \nu = 0, \quad W = \frac{\cot \chi}{R(\tau)}. \quad (\text{E2})$$

Here the left-hand sides are in our notation. (On the right-hand sides,  $\chi$  is a coordinate, not our perturbation variable of the same name.) We also have, from the equation of state  $p = 0$ , that

$$m = \frac{4\pi}{3} r^3 \rho, \quad c_s^2 = 0. \quad (\text{E3})$$

The variable  $U$  of CPM1 is  $-\beta$  in our notation. Our variable  $\Pi$  is related to the interior and exterior variables of CPM1 as

$$\Pi \text{ (our notation)} = -\frac{\pi_1}{l(l+1)R^2 \sin^2 \chi} \text{ (CPM interior)} = -\frac{\tilde{\pi}_1}{l(l+1)r^2} \text{ (CPM exterior)}. \quad (\text{E4})$$

The continuity conditions of CPM1 agree with the continuity of  $\Pi$  that we find.

We first discuss the polar perturbations of Oppenheimer-Snyder in our notation. For homogeneous dust, the interior equations of motion simplify greatly, to

$$-\ddot{\chi} + \chi'' = 3\mu\dot{\chi} + 2W\chi' + \frac{(l+2)(l-1)}{r^2}\chi, \quad (\text{E5})$$

$$-\ddot{k} = U\dot{\chi} + 4U\dot{k} - W\chi' + 2\left(W^2 - \frac{1}{r^2}\right)(\chi + k) - \frac{(l-1)(l+2)}{2r^2}\chi \quad (\text{E6})$$

$$-\dot{\psi} = 2U\psi + \chi'. \quad (\text{E7})$$

$\chi$  therefore obeys an autonomous wave equation describing gravitational waves that do not couple to matter perturbations. The sound wave equation normally obeyed by  $k$  has degenerated into an ODE along matter world lines.  $k$  and  $\psi$  are therefore obtained by solving ODEs after the autonomous wave equation for  $\chi$  has been solved.

We now compare with CPM2. On an Oppenheimer-Snyder interior, our perturbation variable  $\chi$  is essentially the variable  $Q_1$  of [11], and is related to their final variable  $\psi$  as

$$\chi = \sqrt{\frac{5}{4\pi}} Q_1, \quad \chi = \frac{\sin \chi}{R(\tau)} \psi, \quad (\text{E8})$$

where again the notation on the right-hand side is that of CPM2, where  $\chi$  is a coordinate. We find that our equation (E5), restricted to  $l = 2$ , is equivalent to the autonomous wave equation (II-41) for the variable  $\psi$  of CPM2. The exterior variables  $\hat{Q}_1$  and  $\hat{\psi}$  of CPM2, are the same as the ones used by Seidel and given above in (D2). The exterior equation of CPM2 is just the Zerilli equation.

Although CPM2 can and do ignore  $k$  and  $\psi$  in the interior, they need some information about their values at the surface for imposing the matching conditions. CPM2 evolve a variable  $\Delta$  along the stellar surface that is related to our variable  $k$  via

$$k = \frac{1}{3 \sin \chi} \Delta. \quad (\text{E9})$$

Equation (B24) or (II-43c) of [11], an ODE evolution equation for  $\Delta$ , is our Eq. (E6), restricted to the boundary. Equation (II-43a) of [11] agrees with our injection equation for  $k$ , Eq. (91), if one expresses  $Z^i$  through  $Z'$  and  $\dot{Z}$ . CPM2 use it not as an injection equation for  $k$  (as already discussed, they do not evolve  $k$  at all), but as an extraction equation for  $Z$  of the form  $AZ + B\dot{Z} + CZ' = Dk$ , using the value of  $k$  on the boundary that is obtained from (II-43c). The injection equation for  $\chi$  of CPM2 is (II-43b) of [11]. Translated into our notation, is

$$\frac{3\chi}{r} = \ddot{Z} + \frac{4r}{2r+3m}U\dot{Z} + \frac{3m}{2r+3m}\frac{m}{r^3}Z - \frac{9m}{2r+3m}\frac{k}{r}, \quad (\text{E10})$$

where the right-hand side is evaluated just outside the surface, even though the derivatives have been expressed in the fluid frame. When we transform the derivatives on the right-hand side into the radial frame, and then used the Zerilli equation to eliminate  $\hat{\ddot{Z}}$ , we find that this equation agrees with our injection equation (92).

## APPENDIX F: POLAR PERTURBATIONS OF A STATIC STAR: COMPARISON WITH THORNE *ET AL.*

The matching conditions for the  $l \geq 2$  polar perturbations of Thorne I [12], equation (19), are corrected in Thorne II [13], equations (B1a-b), and we compare the corrected version with our results. In the following the left-hand sides are Thorne's notation and the right-hand sides are ours. We use the fact that our gauge-invariant variables are identical with the gauge-dependent ones in Regge-Wheeler gauge, which is the one Thorne *et al.* use.

Equation (D4) of our Paper I contains two sign mistakes: the overall signs of  $\gamma + \psi/2$  and of  $\alpha$  should be reversed. Correcting this, we see that  $e^{-\nu/2}W_{T,t} = r^2(\gamma + \frac{\psi}{2})$ . On a *static* background, this in turn is the frame  $\dot{W}_T$  in our notation. Comparing with our equation (55), we have  $W_T = -r^2N$  (on a static background only), where we have set the integration constant to zero as it does not carry physical information. We also have  $H_0 = H_2 = -(\chi + k)$ ,  $e^{(\lambda+\nu)/2}H_1 = \psi$ , and  $K = -k$ . On the static background,  $e^{-\lambda/2}f_{,r} = f'$ . (Note that Thorne uses  $f'$  for  $f_{,r}$ , while we

use it for the frame derivative along  $n^A$ .)  $e^\lambda$  is continuous because the Hawking mass is continuous. Thorne *et al.* also choose  $e^\nu$  to be continuous. With these correspondences, we find that the five continuous perturbation fields of Thorne II, (B1a-b), are equivalent to the five

$$\chi, \quad k, \quad \psi, \quad \chi', \quad k' + 8\pi\rho N. \quad (\text{F1})$$

These are our continuity conditions, given that  $\mu$  vanishes for a static background.

The special case  $l = 1$  has been covered in Thorne V [14]. Using the relations (D3-D4) of Paper I we have re-obtained all their evolution equations (14) and (15), with the one difference that their (14a) should have an additional term  $-S\gamma_{,r}/\gamma$  (notation of Thorne, see there). We have verified that their regularity conditions (20) at the center agree with our conditions (117-120) of Paper I. The matching equations (21b-e) of Thorne V are equivalent to our matching conditions (69-73), if one restricts the latter to a static background and assumes the gauge choice  $\eta = \hat{\chi} = \hat{\psi} = 0$  made by Thorne V in the exterior.

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